*J. Fluid Mech.* (1983), *vol.* 126, *pp.* 533–544
*Printed in Great Britain*


# The deformation of a viscous particle surrounded by an elastic shell in a general time-dependent linear flow field

By P. O. BRUNN
Department of Chemical Engineering and Applied Chemistry, Columbia University, New York

(Received 28 April 1981 and in revised form 25 June 1982)

The dynamics of a viscous particle surrounded by an elastic shell of arbitrary thickness freely suspended in a general linear flow field is investigated. Assuming the unstressed shell to be spherical, an analysis is presented for the case in which the flow-induced deformation leads to small departures from sphericity. The general time-dependent evolution of shape is derived and various special cases (purely elastic sphere, rigid and gaseous interior, elastic membranes) are discussed in detail. It is found that for steady-state flows the equilibrium deformations are absolutely stable and depend only upon the shell thickness, although the rates at which they are attained show the effect of the inside viscosity, too.

## 1. Introduction

Any non-rigid particle suspended in an immiscible liquid that undergoes any type of non-uniform flow will in general deform from the shape it would assume at rest. Even if the shape at rest is spherical and even if the deformation induced by the flow field is small, only a few models have been extensively investigated. Apart from the liquid-droplet model there are the purely elastic or viscoelastic particles (Roscoe 1967; Goddard & Miller 1967) and the microcapsules (Barthès-Biesel 1980; Guerlet, Barthès-Biesel & Stoltz 1977). The latter term was proposed for viscous particles that are enclosed by an infinitely thin elastic membrane. Natural extensions of that model that come to mind include rheologically complex membranes, non-spherical capsules and membranes of finite thickness. I shall concentrate on the latter case and investigate the small (time-dependent) deformation of a viscous particle surrounded by an elastic shell of arbitrary thickness.

In a recent paper (Brunn 1980) I focused attention on the constitutive equation for a dilute suspension of such particles. For a spherical shell (outer radius $b$, inner radius $a$) the suspension was found to comprise a generalized Maxwell fluid with four time constants. For certain limiting cases less than four characteristic times suffice.

Examples include the purely elastic sphere with just one time constant or the infinitely thin elastic membrane, where only two such constants survive the $(b-a)/b \rightarrow 0$ limit.

To arrive at these results I assumed the particles to remain spherical. Obviously, this cannot be exact since particles of this type will in general deform. Consequently, the solution obtained has to be considered as the zeroth-order result provided that the deviation from spherical shape is sufficiently small. It is the purpose of the present paper to ascertain the conditions for which the deformation is indeed small by studying the actual time-dependent evolution of shape.

More explicitly, we assume the deformation to be of order $\epsilon$, where $\epsilon \ll 1$.† Cox

† It is possible that the outer deformation requires a different $\epsilon$ than the inner deformation.

(1969) has shown in the similar problem of drop-deformation how one can formally expand expressions for dependent variables in the small parameter $\epsilon$ without specifying the content of $\epsilon$ at the outset. As soon as specific results are obtained the actual value of $\epsilon$ can be read off. We shall employ that idea, expecting – on the basis of Cox's (1969) results – that in this way the problem of the determination of the shape in a general time-dependent flow will be no more difficult than the determination of the shape in a simple shear flow.

In §2 we shall formulate the problem mathematically and derive an explicit expression for the slightly deformed spheres (inner and outer). In §3 various special cases (elastic particle, rigid inside, gaseous inside, elastic membrane) will be considered in detail. Finally, in §4 we will consider steady-state flows and show that the equilibrium shapes ultimately obtained are ellipsoids, their main axes coinciding with the principal directions of the rate-of-strain dyadic.

## 2. The evolution of shape

Consider a viscous drop (shear viscosity $\eta_i$) surrounded by an elastic shell (shear modulus $\mu$), which is suspended in a fluid of shear viscosity $\eta_o$. Assume that all materials are incompressible and that in its undeformed or stress-free state the elastic region is a spherical shell of radii $a$ and $b$, $a \leqslant b$. The problem now consists in determining the deformation of the particle when it is placed in an infinite flow field whose (time-dependent) velocity distribution $\mathbf{v}^0$ is prescribed far from the particle. More specifically, we only consider velocity fields that, in the absence of the particle, vary linearly with position, i.e.

$$\mathbf{v}^0 \rightarrow \mathbf{V}^0 = \mathbf{r}\,.\,\mathbf{E}^0(t) + \mathbf{\Omega}(t) \times \mathbf{r} \quad \text{as} \quad r \rightarrow \infty. \tag{2.1}$$

Here, $\mathbf{\Omega}$ is half the vorticity vector, $\mathbf{E}^0$ the 'undisturbed' rate-of-strain dyad, and $\mathbf{r}$ is measured from the particle centre.

Using for the deformed surfaces the expressions

$$r_o = b[1 + \epsilon_b\, g_b(\hat{\mathbf{r}}, t)], \tag{2.2a}$$

$$r_i = a[1 + \epsilon_a\, g_a(\hat{\mathbf{r}}, t)], \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}, \tag{2.2b}$$

we shall assume that the $\epsilon$s, although unknown at this stage, are very small. (Note that to first order in $\epsilon$ we have $r_o = b + u_r|_b$, within an analogous expression for $r_i$.) Previously, it was shown that, in the absence of any inertia effects, the velocity fields inside and outside (termed $\mathbf{v}^{(i)}$ and $\mathbf{v}^{(o)}$, respectively) and the displacement field $\mathbf{u}$ of the elastic region for the undeformed particle can only be of the form (Brunn 1980)†

$$\begin{aligned} \mathbf{v}^{(i)} &= \mathbf{\Omega} \times \mathbf{r} + \mathbf{r}\,.\,\mathbf{E}^{(i)} + \frac{1}{\eta_i}\left[\frac{3}{5} r^2 \mathbf{r}\,.\,\mathbf{C}^{(i)} - \frac{2}{105} r^7 \mathbf{X} : \mathbf{C}^{(i)}\right], \\ \mathbf{u} &= \mathbf{A} \times \mathbf{r}\,.\,\mathbf{r}\,.\left[\frac{3}{5\mu} r^2 \mathbf{C}^{(e)} + \mathbf{E}^{(e)} + \frac{3}{5\mu} r^{-3} \mathbf{c}^{(e)}\right] \\ &\quad + \mathbf{X} : \left[-\frac{2}{105\mu} r^7 \mathbf{C}^{(e)} + \frac{r^2}{10\mu} \mathbf{c}^{(e)} + \mathbf{b}^{(e)}\right], \end{aligned} \tag{2.3a}$$

† The assumption of small deformation allows us to use the concept of linear elasticity, as first approximation.

with†

$$\dot{\mathbf{A}} = \mathbf{\Omega}, \tag{2.3b}$$

$$\mathbf{v}^{(o)} = \mathbf{v}^{(o)} + \frac{3}{5\eta_o} r^{-3}\mathbf{r}\,.\,\mathbf{c}^{(o)} + \mathbf{X} : \left[\frac{r^2}{10\eta_o}\mathbf{c}^{(o)} + \mathbf{b}^{(o)}\right], \tag{2.3c}$$

Here

$$\mathbf{X} = -\frac{\partial^3}{\partial\mathbf{r}\,\partial\mathbf{r}\,\partial\mathbf{r}}(r^{-1}) \tag{2.4}$$

is a symmetric and irreducible tensor of third order. The eight dyads $\mathbf{E}^{(i)}$, $\mathbf{C}^{(i)}$, $\mathbf{E}^{(e)}$, $\mathbf{C}^{(e)}$, $\mathbf{c}^{(e)}$, $\mathbf{b}^{(e)}$, $\mathbf{c}^{(o)}$ and $\mathbf{b}^{(o)}$ are uniquely determined by requiring the stress forces and velocity fields to be continuous across each interface. For the situation at hand, this in itself is an initial-value problem. If we employ a Laplace transform – denoted by an overbar on the corresponding function – it turns out that all dyads are linearly related to $\mathbf{E}^0$, the undisturbed rate-of-strain dyad $\mathbf{E}^0$ acts as 'driving force'). Inserting these results back into (2.3), we get for $\bar{u}_r$, the Laplace transform of the radial component of the displacement field,

$$\bar{u}_r(r = b) = \bar{f}_b\,\bar{\mathbf{E}}^0 : \mathbf{Y}, \tag{2.5a}$$

$$\bar{u}_r(r = a) = \bar{f}_a\,\bar{\mathbf{E}}^0 : \mathbf{Y}, \tag{2.5b}$$

with

$$\mathbf{Y} = \mathbf{Y}(\hat{\mathbf{r}}) = 3\hat{\mathbf{r}}\hat{\mathbf{r}} - \boldsymbol{\delta}. \tag{2.6}$$

Explicitly, putting

$$\bar{f} = \bar{f}(s) = \tau_0 \bar{f}^*(T), \quad T = s\tau_0 \tag{2.7}$$

with

$$\tau_0 = \frac{\eta_o}{u} \tag{2.8}$$

a characteristic time and $s$ the ordinary Laplace variable, we have

$$\bar{f}_b^*(T) = b\frac{\frac{5}{9}P_4T^3 + a_2T^2 + a_1T + a_0}{D(T)} \equiv b\frac{N_b(T)}{D(T)}, \tag{2.9a}$$

$$\bar{f}_a^*(T) = a\frac{A_2T^2 + A_1T + A_0}{D} \equiv a\frac{N_a(T)}{D(T)}, \tag{2.9b}$$

with

$$D(T) = P_4T^4 + P_3T^3 + P_2T^2 + P_1T + P_0. \tag{2.10}$$

All coefficients $P_i$, $a_i$ and $A_i$ are, for fixed value of the viscosity ratio

$$\lambda = \frac{\eta_o}{\eta_i}, \tag{2.11a}$$

tenth-order polynomials in the shell thickness ratio

$$x = \frac{b}{a}, \tag{2.11b}$$

while they are second-order polynomials in $\lambda$ if $x$ is fixed. It will prove convenient to display the $\lambda$-dependence explicitly by putting

$$P_2 = P_2(x, \lambda) = P_{20} + \lambda P_{21} + \lambda^2 P_{22}, \tag{2.12}$$

† Note that this result implies that the reference configuration, relative to which the displacement $\mathbf{u}$ is defined, rotates with the fluid. Since the elastic region is uniform, the identity of a material point passing through a given position at a given time is irrelevant. Consequently, the solid-body rotation of the shell can be subtracted out of the problem.

where the $P_{2i}$ are only dependent upon $x$ (tenth-order polynomials). Analogous definitions will be used for all the coefficients (see appendix).

To first order in the small deformation parameter $\epsilon$ the radial component of the displacement field determines the shape of the deformed sphere (with $r_o = b+u_r|_b$ to first order in $\epsilon$, with an analogous expression for $r_i$). Thus, using (2.5) and (2.7), we get by recalling (2.2)

$$\epsilon_b g_b = \int_0^t dt' f_b\left(\frac{t-t'}{\tau_0}\right) \mathbf{E}^0(t') : \mathbf{Y}(\hat{\mathbf{r}}), \tag{2.13a}$$

and similarly

$$\epsilon_b g_b(\hat{\mathbf{r}}, t) = \int_0^t dt f_a\left(\frac{t-t'}{\tau_0}\right) \mathbf{E}^0(t') : \mathbf{Y}(\hat{\mathbf{r}}). \tag{2.13b}$$

Here $f_a$ and $f_b$ denote the inverse transforms of $\bar{f}_a^*/a$ and $\bar{f}_b^*/b$, respectively. As shown before (Brunn 1980), the four roots $T_j$ of $D(T)$ are real, non-positive and, in general, distinct.† Consequently, putting

$$\epsilon_b g_b(\hat{\mathbf{r}}, t) = \epsilon_b \mathbf{G}_b(t) : \mathbf{Y}(\hat{\mathbf{r}}), \tag{2.14}$$

with a similar definition for $\mathbf{G}_a(t)$, we get

$$\epsilon_b \mathbf{G}_b(t) = \sum_{j=1}^{4} \frac{N_b(T_j)}{D'(T_j)} \int_0^t dt' \exp\left[\frac{T_j}{\tau_0}(t-t')\right] \mathbf{E}^0(t'), \tag{2.15a}$$

and

$$\epsilon_a \mathbf{G}_a(t) = \sum_{j=1}^{4} \frac{N_a(T_j)}{D'(T_j)} \int_0^t dt' \exp\left[\frac{T_j}{\tau_0}(t-t')\right] \mathbf{E}^0(t'). \tag{2.15b}$$

Here, $D'(T_j)$ is the derivative of $D$ with respect to $T$ evaluated at $T_j$.

Equations (2.14) and (2.15) are the desired results. They represent ellipsoids whose orientation of the major and minor axes will, in general, not coincide with the instantaneous principal directions of $\mathbf{E}^0(t)$. Rather the whole history of the rate-of-strain dyad matters. The details of the particle–fluid system (i.e. $\lambda$ and $x$) are contained in the (four) characteristic times

$$\tau_j = -\frac{\tau_0}{T_j}. \tag{2.16}$$

Some specific examples shall suffice to illustrate this point.

## 3. Examples

### 3.1. *The elastic sphere*

For a particle consisting solely of elastic material we have $a = 0$. Equivalently, this requires $x \to \infty$, and in this limit $\bar{f}_b^*$ assumes the particularly simple form

$$\bar{f}_b^*(T) = \tfrac{5}{9} b \frac{1}{T+\frac{2}{3}} \quad \text{for} \quad x \to \infty. \tag{3.1}$$

This implies

$$\epsilon_b \mathbf{G}_b = \frac{5}{9} \int_0^t dt' \exp\left[-\frac{2}{3\tau_0}(t-t')\right] \mathbf{E}^0(t'), \tag{3.2}$$

so that only one characteristic time $3\eta_0/2\mu$ has to be reckoned with. The sole importance of this time constant has been realized before (Goddard & Miller 1967; Roscoe 1967).

† If a multiple root occurs this root is also a root of the numerator (see §3.2).

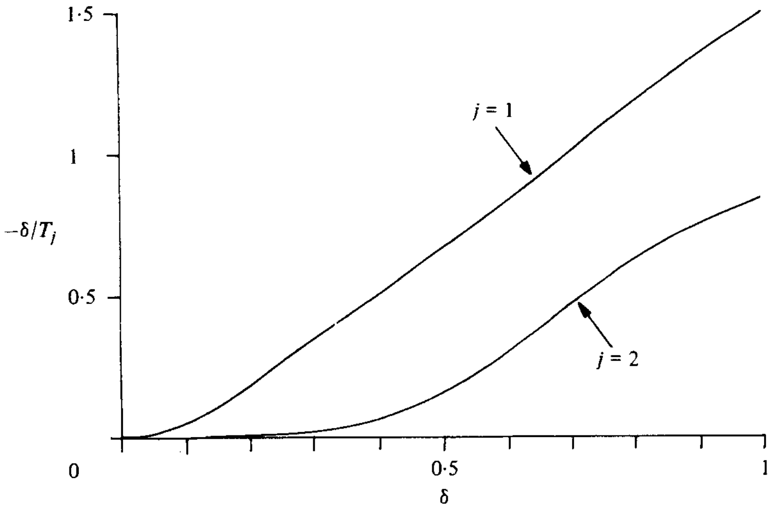


FIGURE 1. The roots $T_j$ of $D_0$ for a rigid interior as functions of the shell thickness $\delta$.

### 3.2. *Rigid interior*

Mathematically we can view a rigid interior as an infinitely viscous fluid. Thus, taking the $\lambda \rightarrow 0$ limit, the result

$$\bar{f}_a^* = 0 \quad \text{for} \quad \lambda = 0 \tag{3.3}$$

follows at once. Actually, we could have anticipated that relation since the inside can not deform at all. On the other hand

$$\bar{f}_b^*(T) = b\frac{\frac{5}{9}P_4T + a_{20}}{P_4T^2 + P_{30}T + P_{20}} \equiv b\frac{N_{b0}(T)}{D_0} \tag{3.4}$$

implies that the sum in (2.15*a*) contains only two terms.† Introducing the (dimensionless) shell thickness

$$\delta \equiv \frac{b-a}{b} = 1 - \frac{1}{x}, \tag{3.5}$$

a glance at figure 1 reveals that the roots $T_j$ increase as the shell thickness decreases. As a matter of fact for $\delta < 0{\cdot}3$ the root $T_2$ is so large that we may approximate it by infinity. With this understanding we have for $\delta < 0{\cdot}3$

$$\epsilon_b \mathbf{G}_b(t) = \frac{P_4[\frac{5}{9}P_{20} + a_{20}T_1]}{P_{20}Q}\tau_0 \mathbf{E}^0(t) + \left\{\frac{5}{18} - \frac{\frac{5}{18}P_{30} - a_{20}}{Q}\right\}\int_0^t dt' \exp\left[\frac{T_1}{\tau_0}(t-t')\right]\mathbf{E}^0(t'),$$

with

$$Q = [P_{30}^2 - 4P_{20}P_4]^{\frac{1}{2}}, \quad T_1 = \frac{1}{2P_4}\left[-P_{30} + Q\right]. \tag{3.6}$$

If the thickness of the shell decreases further, say to less than $0{\cdot}03$, we are justified in taking the $T_1 \rightarrow \infty$ limit, too. Although formally this leads to

$$\lim_{T_1 \rightarrow \infty} \epsilon_b \mathbf{G}_b(t) = \frac{a_{20}}{P_{20}}\tau_0 \mathbf{E}^0(t), \tag{3.7}$$

the actual value of $a_{20}/P_{20}$ at such small shell thicknesses is zero (see also figure 4). What this means is that an elastic shell surrounding a rigid sphere has to be sufficiently thick before it will show any deformation. This, however, was to be expected.

† The double root of $D$ for $T = 0$ is also a double zero of $N_b$.

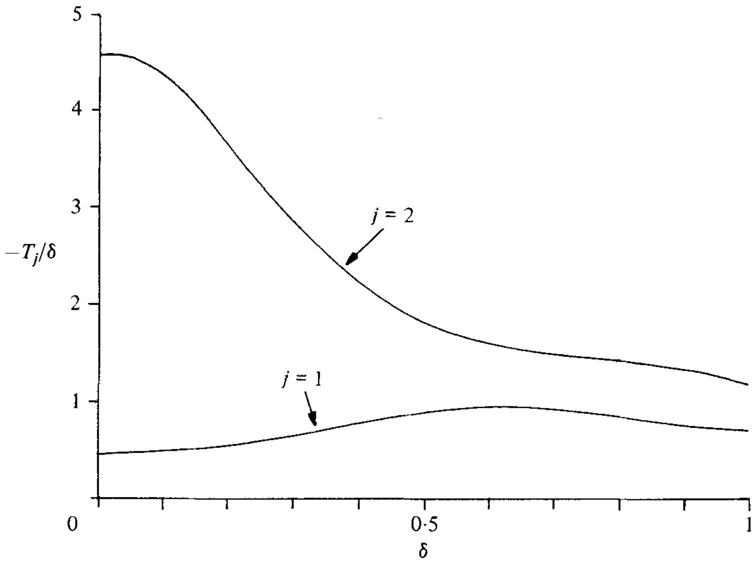


FIGURE 2. The roots $T_i$ of $D_\infty$ for a gaseous interior as functions of the shell thickness $\delta$.

At first sight it may seem strange that we have plotted $T_i/\delta$ instead of $T_i$. The reason is that we want to include all shell thicknesses. For infinitely thin membranes one uses membrane stresses that require the quantity $(b-a)\mu$ instead of $\mu$. A balance between viscous and elastic forces at the membrane surface reveals then that instead of $\tau_0$ one needs $\tau_0/\delta$. Recalling (2.15), we thus have to know the behaviour of $T_j/\delta$ as $\delta$ approaches zero. For the case at hand $T_1/\delta$ and $T_2/\delta$ blow up like $\delta^{-2}$ and $\delta^{-4}$ respectively. Although $T_j/\delta$ is needed only for $\delta \to 0$ we shall always use it. For given, but non-zero, $\delta$, this means that we are merely scaling the roots $T_j$.

### 3.3. *The gaseous interior*

Here the limit $\lambda \to \infty$ is appropriate, and the results

$$\tilde{f}_b^* = b\frac{a_{12}T + a_{02}}{P_{22}T^2 + P_{12}T + P_{02}} \equiv b\frac{N_{b\infty}}{D_\infty}, \tag{3.8a}$$

$$\tilde{f}_a^* = a\frac{A_{12}T + A_{02}}{P_{22}T^2 + P_{12}T + P_{02}} \equiv b\frac{N_{a\infty}}{D_\infty} \tag{3.8b}$$

follow readily. Thus again only two roots

$$T_{1,2} = \frac{1}{2P_{22}}\left[-P_{12} \pm \left(P_{12}^2 - 4P_{22}P_{02}\right)^{\frac{1}{2}}\right] \tag{3.9}$$

have to be reckoned with. As can be seen from figure 2, the roots are never very large over the whole domain of $\delta$ with

$$T_1/\delta = -0{\cdot}438 \quad \text{for} \quad \delta \to 0, \quad T_1/\delta = -\tfrac{2}{3} \quad \text{for} \quad \delta \to 1,$$
$$T_2/\delta = -4{\cdot}56 \quad \text{for} \quad \delta \to 0, \quad T_1/\delta = -\tfrac{19}{16} \quad \text{for} \quad \delta \to 1$$

as limiting values. Thus, apart from certain additional simplifications to be discussed below, (2.15) have to be used as they stand, although only two terms appear in the sum.

### 3.4. *The membrane approximation*

If the elastic shell is very thin we put $\delta \ll 1$. Excluding the cases $\lambda = 0$ and $\lambda = \infty$, which have already been treated, it is readily checked that in this limit two ($T_1$, $T_2$) of the four roots $T_j$ of $D$ are proportional to $\delta$, one ($T_3$) is proportional to $\delta^{-1}$, while $T_4$ is proportional to $\delta^{-3}$. As long as we admit flow fields that may vary in time arbitrarily rapidly, all four roots have to be kept. Excluding such exceptional flow fields offers the opportunity to approximate $T_3$ and $T_4$ by infinity. After all, as seen from the numerical results obtained before (Brunn 1980), we get for $\lambda = 0{\cdot}2$ and $\delta = 0{\cdot}01$ the ratios $T_1:T_2:T_3:T_4 = 1:8:31{\cdot}8\times10^3:50{\cdot}5\times10^7$, while for the same $\lambda$ but at a shell thickness of $\delta = 0{\cdot}002$ we have $T_1:T_2:T_3:T_4 = 1:8:7{\cdot}8\times10^5:30{\cdot}5\times10^{10}$. Thus, taking the limit $T_{3,4}' \to \infty$,† we eventually obtain

$$\epsilon_b \mathbf{G}_b = \epsilon_a \mathbf{G}_a = C_1 \int_0^t dt' \exp\left[\frac{T_1}{\tau_0}(t-t')\right] \mathbf{E}^0(t') + C_2 \int_0^t dt' \exp\left[\frac{T_2}{\tau_0}(t-t')\right] \mathbf{E}^0(t') \quad (3.10)$$

with

$$C_{1,2} = \frac{5}{12}\frac{\lambda}{1+\frac{3}{2}\lambda}\left\{1 \pm \frac{65}{228}\frac{1+\frac{24}{13}\lambda}{\Delta}\right\}, \quad (3.11a)$$

$$\frac{T_{\frac{1}{2}}}{\delta} = \frac{6\lambda}{(1+\frac{16}{19}\lambda)(1+\frac{3}{2}\lambda)}\{-\tfrac{5}{12}(1+\tfrac{24}{19}\lambda) \pm \Delta\}, \quad (3.11b)$$

$$\Delta = \tfrac{1}{12}(\tfrac{283}{19})^{\frac{1}{2}}\{1+\tfrac{14256}{5377}\lambda + \tfrac{9792}{5377}\lambda^2\}^{\frac{1}{2}}. \quad (3.11c)$$

Equation (3.10) (which, by relying on a membrane approximation from the outset, agrees with Barthès-Biesel & Rallison 1981) implies that for a sufficiently thin shell the time evolution of the outside surface exactly equals the time evolution of the inside surface. But, as stated before for a shell of given (although small) thickness, there are upper limits on the time variation of the flow field beyond which this result will fail. Phrased differently, (3.10) should be the first-order small-deformation result of an elastic membrane (such that the membrane equations apply). If a thin shell is classified as a membrane for one flow field, a much thinner shell may be needed in order for the shell to be called a membrane for a more rapidly varying flow field. For example, in oscillatory flow of frequency $w$, the relaxation time $\tau_3$ (besides $\tau_1$ and $\tau_2$) would have to be taken account of if $|w\tau_0/\delta|$ is not negligible in comparison to $\delta^{-2}$. Only an infinitely thin shell (no thickness) constitutes a perfect membrane (two relaxation times).

## 4. Steady-state flows

If the flow field is independent of time the integrations (2.15) are readily performed. Alternatively, we can employ the final-value theorem for a Laplace transform. Since the $T_j$ are real and negative, the equilibrium deformations, which are asymptotically attained, are

$$\epsilon_b \mathbf{G}_b = \left\{\begin{matrix} \tau_0 \dfrac{a_{20}}{P_{20}} \mathbf{E}^0 & (\lambda = 0) \\ \tau_0 \dfrac{a_{02}}{P_{02}} \mathbf{E}^0 & (\text{otherwise}) \end{matrix}\right\}, \quad (4.1a)$$

† Strangely enough there is no contribution from these limits. This contrasts with our previous study, where we found that these limits had to be taken carefully, since they made non-vanishing contributions to the rheological equation of state.

and
$$\epsilon_a \mathbf{G}_a = \begin{Bmatrix} 0 & (\lambda = 0) \\ \tau_0 \dfrac{A_{02}}{P_{02}} \mathbf{E}^0 & (\text{otherwise}) \end{Bmatrix}. \tag{4.1b}$$

Thus the outer and inner surfaces each deform into an ellipsoid whose major and minor axes coincide with the principal directions of $\mathbf{E}^0$. Since $a_{02}/P_{02}$ as well as $A_{02}/P_{02}$ depend only upon $x$ (or $\delta$) the steady-state deformation is independent of $\lambda$ for $\lambda \neq 0$. This fact, which for a membrane was first pointed out by Guerlet *et al.* (1977), is remarkable, for the transient terms leading to the stable equilibrium deformation do depend upon $\lambda$.

Without loss of generality we will concentrate on the important case of two-dimensional motions, which can always be expressed as

$$V_x = Gy, \quad V_y = \alpha Gx \quad (-1 \leqslant \alpha \leqslant 1). \tag{4.2}$$

Here $G$ denotes the shear rate. Varying the parameter $\alpha$ from $-1$ to zero and finally up to 1 produces all homogeneous two-dimensional flow fields from pure rotation ($\alpha = -1$), to pure shear ($\alpha = 0$) and finally pure straining motion ($\alpha = 1$). If the angle $\phi$ is measured relative to the $x$-axis, the deformed surfaces are represented by

$$r_0(\phi) = b\{1 + D_b \sin 2\phi\}, \tag{4.3a}$$

$$r_i(\phi) = a\{1 + D_a \sin 2\phi\}. \tag{4.3b}$$

Here $D_b$ and $D_a$ are indications of the corresponding deformations. Their geometrical significance becomes apparent if we write

$$D_b = \frac{L_b - B_b}{L_B + B_b}, \tag{4.4}$$

where $L_b$ is the longest and $B_b$ the shortest axis of the ellipsoid. An analogous expression holds for $D_a$. Putting

$$D_b = (1+\alpha)\frac{\tau_0}{\delta} G D_b^*, \tag{4.5}$$

and similarly introducing $D_a^*$, we have explicitly

$$D_b^* = \begin{Bmatrix} \dfrac{3}{2}\dfrac{a_{20}}{P_{20}}\delta & (\lambda = 0) \\ \dfrac{3}{2}\dfrac{a_{02}}{P_{02}}\delta & (\text{otherwise}) \end{Bmatrix}, \tag{4.6a}$$

$$D_a^* = \begin{Bmatrix} 0 & (\lambda = 0) \\ \dfrac{3}{2}\dfrac{A_{02}}{P_{02}}\delta & (\text{otherwise}) \end{Bmatrix}. \tag{4.6b}$$

Since $D_b^*$ and $D_a^*$ are both independent of $\lambda$, for $\lambda \neq 0$, figure 3 is characteristic for all particles, for which $b = 1$, $a = \frac{1}{2}$ and $(1+\alpha)\tau_0 G = 0{\cdot}1$.

In figure 4 the functions $D_a^*$ and $D_b^*$ have been plotted. The exceptional case where the inside is rigid is shown in figure 4(*a*). In this case, $D_b^*$ vanishes like $\delta^3$ as $\delta \to 0$. Consequently, $\delta$ has to be sufficiently thick ($\delta \geqslant 0{\cdot}12$) in order for any deformation of the outside to occur. Basically this confirms our earlier result of §3.2. Furthermore, in this situation an increase in the thickness of the elastic region leads to an

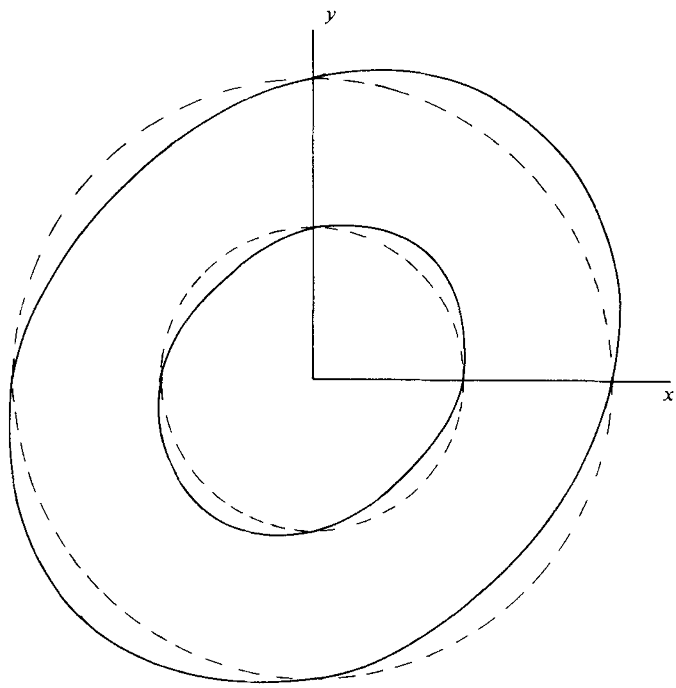


FIGURE 3. The first-order deformation of the shell $b = 1$, $a = \frac{1}{2}$ ($\delta = \frac{1}{2}$) in a weak two-dimensional flow $(1+\alpha)G\tau_0/\delta = 0{\cdot}1$. The dotted curves represent the unstressed shell.

increasingly larger deformation. The final limiting value of $\frac{5}{4}$ for $D_b^*$ as $\delta \rightarrow 1$, i.e. for the purely elastic particle, has already been reported (Roscoe 1967).

As shown in figure 4(*b*), for all non-zero values of $\lambda$ no monotonic behaviour of $D_a^*$ or $D_b^*$ is found. Only at $\delta = 0$ does $D_a^*$ assume the same value as $D_b^*$, namely $\frac{25}{12}$. But $\delta = 0$ corresponds to an infinitely thin membrane. Thus, it is no surprise to find that in this case the theory of membranes furnishes exactly the same result (Guerlet *et al.* 1977).

For all other values of $\delta$, the function $D_b^*$ is strictly less than $D_a^*$. Actually, we should not compare $D_b^*$ with $D_a^*$, but rather $D_b^*$ with $(1-\delta)\,D_a^*$.† From this we see that starting from an infinitely thin membrane ($\delta = 0$) the actual deformation $r_0 - b$ of the outer surface essentially coincides with the actual deformation $r_i - a$ of the inner surface up to a shell thickness of $\delta = 0{\cdot}3$. Beyond that thickness, the outer surface deforms more than the inner surface. Taking formally the limit of a purely elastic particle furnishes the maximum difference (the limiting values are 0 and $\frac{5}{4}$ respectively).

Before leaving this subject, it seems worthwhile to point out that the results of that chapter could also have been obtained by using for $\mathbf{v}^{(0)}$ the flow field corresponding to homogeneous flow around a freely rotating rigid sphere. The steady-state deformations of the shell then result by invoking a stress balance across each fluid–solid interface. This approach, which was successfully employed by Barthès-Biesel (1980) for an infinitely thin membrane (even up to the $O(\epsilon^2)$ deformation) is far simpler, but furnishes no clues to the stability of the equilibrium solution. It does demonstrate, however, why the steady-state results have to be independent of $\lambda$.

† Note that the term $b/(b-a)$ is required for the outer surface as $\delta \rightarrow 0$, while $a/(b-a)$ is the corresponding term for the inner surface.

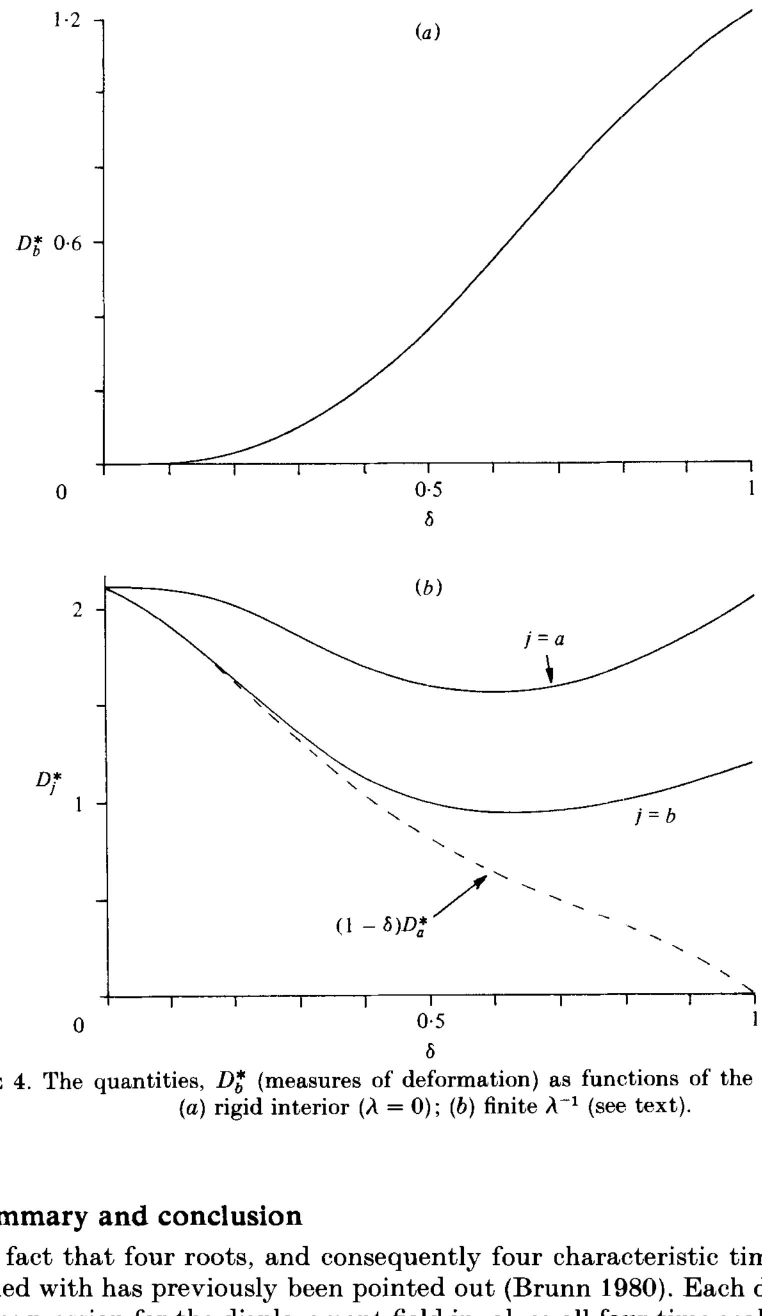


FIGURE 4. The quantities, $D_b^*$ (measures of deformation) as functions of the shell thickness $\delta$: (a) rigid interior ($\lambda = 0$); (b) finite $\lambda^{-1}$ (see text).

## 5. Summary and conclusion

The fact that four roots, and consequently four characteristic times, have to be reckoned with has previously been pointed out (Brunn 1980). Each dyad appearing in the expression for the displacement field involves all four time scales. Associating a given characteristic time with a particular mode of deformation may not be possible, especially since each dyad shows a different evolution in time. Only the overall deformation of the shell, involving the combination

$$\frac{3}{7\mu} r^3 \mathbf{C}^{\mathrm{e}} + r\mathbf{E}^{\mathrm{e}} + \frac{3}{2\mu r^2}\, \mathbf{c}^{\mathrm{e}} + \frac{9}{r^4}\, \mathbf{b}^{\mathrm{e}},$$

has been studied in this paper.

According to our basic equations (2.13) and (2.15), the first-order deformations of the inner and outer surfaces are of the form

$$\epsilon_b g_b = \tau_0 E F_b(\lambda, \delta; t), \tag{5.1a}$$

$$\epsilon_a g_a = \tau_0 E F_a(\lambda, \delta; t). \tag{5.1b}$$

Here $E$ is some measure of the maximum value of the rate-of-strain dyad of the undisturbed flow, and the functions $F_b$ and $F_a$ have explicitly been obtained for a number of special cases. By comparing (5.1) with (2.15) it becomes apparent that (i) the maximum deformation requires the $t \rightarrow \infty$ limit for $F_b$ and $F_a$ (note that the roots $T_j$ are real and non-positive), and that (ii) this result (which is the steady-state limit) is always stable. This leads to

$$\lim_{t \rightarrow \infty} F_b(\lambda, \delta, t) = \frac{a_{02}}{P_{02}}, \tag{5.2a}$$

$$\lim_{t \rightarrow \infty} F_a(\lambda, \delta, t) = \frac{A_{02}}{P_{02}}, \tag{5.2b}$$

provided that $\lambda^{-1}$ is finite. The functions $a_{02}/P_{02}$ and $A_{02}/P_{02}$, depending only upon the shell thickness $\delta$, have been extensively studied in §4. This leads us to conclude that the deformations investigated in this study will indeed be small, provided that the shear modulus $\mu$ is large, the value of $E$ is small, or both (weak flows).

Since $\mu$ large corresponds to a nearly rigid membrane, the alternative method of solution presented at the end of §4 is physically obvious.

There is one case in which $E$ need not be small. This concerns the case for which the interior is extremely viscous, $\lambda \approx 0$. For $\lambda = 0$ the right-hand sides of (5.2*a, b*) have to be replaced by $a_{20}/P_{20}$ and 0 respectively. Although $a_{20}/P_{20}$ is strictly non-negative it tends to zero like $\delta^2$ for $\delta \rightarrow 0$. Thus we expect small deformations for sufficiently thin shells with a very viscous interior, even though the strength of the flow field may be unlimited. On the other hand, in order for both the outer and inner deformations of a shell of finite thickness to be small, a weak flow field is always needed.

Finally, we note that, by using the $O(1)$ velocity fields, the $O(\epsilon)$ deformation can be calculated (as demonstrated). With the deformed shape thus specified, the $O(\epsilon)$ fluid problem can be attacked analogously to yield eventually information about the $O(\epsilon^2)$ deformation. While strictly speaking this is true, two inherent difficulties have to be solved. The first one, already present at $O(1)$, has to do with the interaction between a fluid and a deformable solid and the different reference frames customarily used: Eulerian for the fluid and Lagrangian for the solid. This means that the velocity of each elastic interface has to be expressed in terms of the coordinates appropriate for the frame chosen. At $O(\epsilon)$, where the problem is highly nonlinear, this is a non-trivial task. Secondly, a deformation of $O(\epsilon^2)$ can in general no longer be described by means of the linear stress–strain relation used in this paper. To tackle such problems, a more-general framework is needed.

## Appendix. The coefficients $P_i$, $a_i$ and $A_i$ appearing in the functions $D$, $N_b$ and $N_a$

Using the definition (2.12) to extract the $\lambda$-dependence, we have explicitly

$$P_4 = P_{40} = 2x^{10} - \tfrac{25}{2}x^7 + 21x^5 - \tfrac{25}{2}x^3 + 2, \tag{A 1}$$

$$P_{30} = \tfrac{89}{24}x^{10} + \tfrac{25}{8}x^7 - 7x^5 + \tfrac{25}{6}x^3 - 4, \tag{A 2a}$$

$$P_{31} = \tfrac{89}{19}x^{10} + \tfrac{75}{38}x^7 - \tfrac{63}{19}x^5 + \tfrac{25}{38}x^3 - 4, \tag{A 2b}$$

$$P_{20} = \tfrac{19}{12}x^{10} + \tfrac{75}{8}x^7 - 14x^5 + \tfrac{25}{3}x^3 + 2, \tag{A 3a}$$

$$P_{21} = \tfrac{7921}{912}x^{10} - \tfrac{75}{152}x^7 + \tfrac{21}{19}x^5 - \tfrac{25}{114}x^3 + 8, \tag{A 3b}$$

$$P_{22} = \tfrac{48}{19}x^{10} + \tfrac{200}{19}x^7 - \tfrac{336}{19}x^5 + \tfrac{225}{19}x^3 + 2, \tag{A 3c}$$

$$P_{11} = \tfrac{89}{24}x^{10} - \tfrac{225}{152}x^7 + \tfrac{42}{19}x^5 - \tfrac{25}{57}x^3 - 4, \tag{A 4a}$$

$$P_{12} = \tfrac{89}{19}x^{10} - \tfrac{50}{19}x^7 + \tfrac{112}{19}x^5 - \tfrac{75}{19}x^3 - 4, \tag{A 4b}$$

$$P_{02} = 2x^{10} - \tfrac{150}{19}x^7 + \tfrac{224}{19}x^5 - \tfrac{150}{19}x^3 + 2, \tag{A 5}$$

$$a_{20} = \tfrac{5}{3}[\tfrac{19}{24}x^{10} + \tfrac{25}{48}x^7 - \tfrac{77}{16}x^5 + \tfrac{25}{6}x^3 - \tfrac{2}{3}], \tag{A 6a}$$

$$a_{21} = \tfrac{5}{9}[\tfrac{89}{19}x^{10} + \tfrac{75}{38}x^7 - \tfrac{63}{19}x^5 + \tfrac{25}{38}x^3 - 4], \tag{A 6b}$$

$$a_{11} = \tfrac{5}{3}[\tfrac{89}{48}x^{10} - \tfrac{25}{304}x^7 + \tfrac{231}{304}x^5 - \tfrac{25}{114}x^3 + \tfrac{4}{3}], \tag{A 7a}$$

$$a_{12} = \tfrac{5}{9}P_{22}, \tag{A 7b}$$

$$a_{02} = \tfrac{5}{3}[x^{10} - \tfrac{25}{57}x^7 + \tfrac{77}{19}x^5 - \tfrac{75}{19}x^3 - \tfrac{2}{3}], \tag{A 8}$$

$$A_{21} = \tfrac{25}{3}x^3[\tfrac{1}{3}x^7 + \tfrac{7}{19}x^2 - \tfrac{40}{57}], \tag{A 9}$$

$$A_{11} = \tfrac{25}{3}x^3[\tfrac{19}{48}x^7 - \tfrac{7}{19}x^2 + \tfrac{40}{57}], \tag{A 10a}$$

$$A_{12} = \tfrac{25}{3}x^3[\tfrac{16}{57}x^7 - \tfrac{7}{19}x^2 + \tfrac{40}{57}], \tag{A 10b}$$

$$A_{02} = A_{21}. \tag{A 11}$$

## REFERENCES


BARTHÈS-BIESEL, D. 1980 Motion of a spherical microcapsule freely suspended in a linear shear flow. *J. Fluid Mech.* **100**, 831–858.

BARTHÈS-BIESEL, D. & RALLISON, J. M. 1981 The time-dependent deformation of a capsule freely suspended in a linear shear flow. *J. Fluid Mech.* **113**, 251–267.

BRUNN, P. O. 1980 On the rheology of viscous drops surrounded by an elastic shell. *Biorheol.* **17**, 419–430.

COX, R. G. 1969 The deformation of a drop in a general time-dependent fluid flow. *J. Fluid Mech.* **37**, 601–623.

GODDARD, J. D. & MILLER, C. 1967 Nonlinear effects in the rheology of dilute suspensions. *J. Fluid Mech.* **28**, 657–673.

GUERLET, B., BARTHÈS-BIESEL, D. & STOLTZ, J. F. 1977 Deformation of a sphered red blood cell freely suspended in a simple shear flow. *INSERIM* **71**, 257–264.

FRÖHLICH, H. & SACK, R. 1946 Theory of the rheological properties of dispersions. *Proc. R. Soc. Lond.* A **185**, 415–430.

ROSCOE, R. 1967 On the rheology of viscoelastic spheres in a viscous fluid. *J. Fluid Mech.* **28**, 273–293.